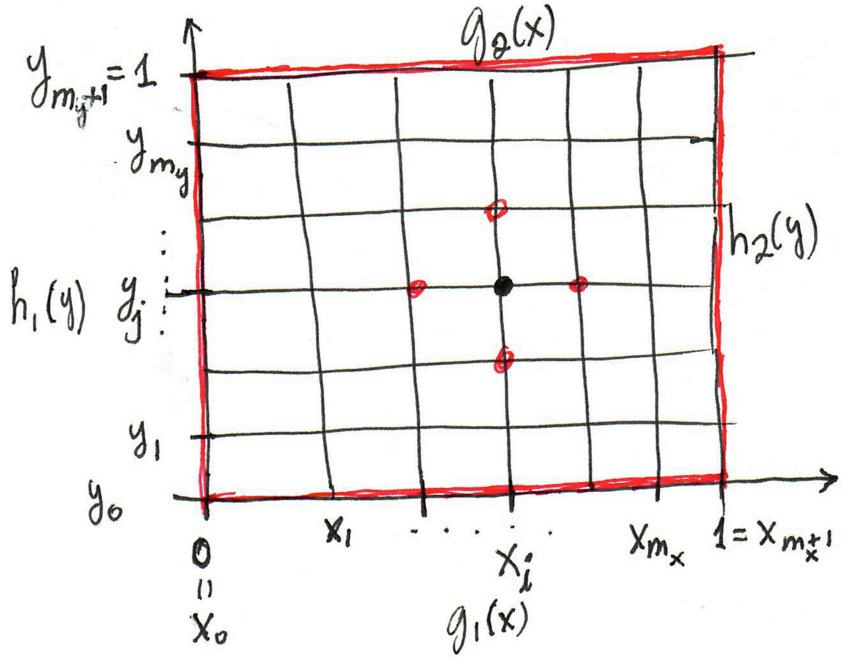
3.2 The 5-point stencil for Laplacian.

Consider a Dirichlet BVP for the Poisson equation in 2D.

$$\begin{cases} \nabla^2 u = u_{xx} + u_{yy} = f(x,y) & & (2.1) \\ u(x,0) = g_1(x), & 0 \le x \le 1 & (2.2) \\ u(x,1) = g_2(x), & 0 \le x \le 1 & (2.3) \\ u(0,y) = h_1(y), & 0 \le y \le 1 & (2.4) \\ u(1,y) = h_2(y), & 0 \le y \le 1 & (2.5) \end{cases}$$

We want to find the solution inside the unit square subject to the boundary conditions (2.2)–(2.5).



$$h_x = \frac{1}{m_x+1}$$

$$h_y = \frac{1}{m_y+1}$$

Let $v_{ij} \simeq u(x_i, y_j)$

Using Centered difference in x and y.

(2.1) Can be approximated by the discrete equation:

$$\boxed{\frac{1}{h_x^2}\left[v_{i-1,j} - 2v_{ij} + v_{i+1,j}\right] + \frac{1}{h_y^2}\left[v_{i,j-1} - 2v_{i,j} + v_{i,j+1}\right] = f_{ij}}$$

$i = 1, 2, \ldots m_x$

$j = 1, 2, \ldots \ldots m_y$

With a truncation error: (3.1)

$$\boxed{\tau_{i,j} = \frac{1}{12} h_x^2\, u_{xxxx}(x_i, y_j) + \frac{1}{12} h_y^2\, u_{yyyy}(x_i, y_j) + O(h_x^4) + O(h_y^4)}$$

Equation (3.1) Can be written as (multiplying by $h_x^2 h_y^2$)

$$h_y^2\left[v_{i-1,j} - 2v_{ij} + v_{i+1,j}\right] + h_x^2\left[v_{i,j-1} - 2v_{ij} + v_{i,j+1}\right] = h_y^2 h_x^2 f_{ij}$$

or

$$h_y^2\left(v_{i-1,j} + v_{i+1,j}\right) + h_x^2\left(v_{i,j-1} + v_{i,j+1}\right) - 2(h_y^2 + h_x^2)v_{i,j} = h_y^2 h_x^2 f_{ij}$$

Dividing by $2(h_y^2 + h_x^2)$

$$\frac{h_y^2}{2(h_y^2+h_x^2)}\left(v_{i-1,j} + v_{i+1,j}\right) + \frac{h_x^2}{2(h_y^2+h_x^2)}\left(v_{i,j-1} + v_{i,j+1}\right) - v_{i,j} = \frac{h_y^2 h_x^2}{2(h_y^2+h_x^2)} f_{ij}$$

Calling $\theta_x \equiv \dfrac{h_y^2}{2(h_y^2+h_x^2)}$, $\theta_y \equiv \dfrac{h_x^2}{2(h_y^2+h_x^2)}$

$$\theta_f \equiv \frac{h_y^2 h_x^2}{2(h_y^2+h_x^2)}$$

Then,

$$\theta_x\left(v_{i-1,j} + v_{i+1,j}\right) + \theta_y\left(v_{i,j-1} + v_{i,j+1}\right) - v_{ij} = \theta_f f_{ij} \tag{4.1}$$

or

$$\theta_y v_{i,j-1} + \theta_x v_{i-1,j} - v_{ij} + \theta_x v_{i+1,j} + \theta_y v_{i,j+1} = \theta_f f_{ij} \tag{4.2}$$

Comparison of (4.2) with (3.10) Leveque.

$$\boxed{\theta_y U_{i,j-1} + \theta_x U_{i-1,j} - U_{ij} + \theta_x U_{i+1,j} + \theta_y U_{i,j+1} = \theta_f f_{ij}} \quad (4.2)$$

$$\boxed{\frac{1}{h^2}\left(U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1} - 4U_{ij}\right) = f_{ij}} \quad (3.10) \text{ Leveque}$$

$$\theta_y = \frac{h_x^2}{2(h_y^2+h_x^2)}, \quad \theta_x = \frac{h_y^2}{2(h_y^2+h_x^2)}, \quad \theta_f = \frac{h_y^2 h_x^2}{2(h_x^2+h_y^2)}$$

If $h_x = h = h_y$

Then, $\theta_y = \frac{1}{4} = \theta_x$ and $\theta_f = \frac{h^2}{4}$. Therefore, (4.2) reduces to

$$\frac{1}{4}\left(U_{i,j-1} + U_{i-1,j} + U_{i+1,j} + U_{i,j+1}\right) - U_{ij} = \frac{h^2}{4} f_{ij}$$

Multiplying this equation by $\frac{4}{h^2}$ (3.10) is obtained.

3.2 Local Truncation Error for the 5-point scheme for Laplacian.

We will based our derivation of the local truncation error for the 5-point scheme on the local truncation error for the centered difference approximation of the second derivative $v''(x)$.

$$\boxed{\frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} = v_i'' + \frac{h^2}{12} v^{(4)}(x_i) + O(h^4)}$$

Continuous Equation

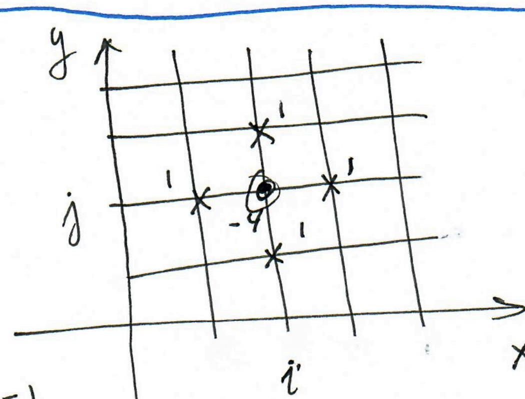
$$\nabla^2 u = 0 \quad (C.E.)$$

Discrete Equation

$$\nabla_5^2 U_{ij} = 0 \quad (D.E.)$$

where

$$\boxed{\nabla_5^2 U_{ij} = \frac{1}{h^2}\left[U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1} - 4U_{ij}\right]}$$



The local truncation error, τ_{ij} for this scheme is obtained replacing the exact solution $u(x,y)$ into (D.E) and subtracting (C.E) from it.

In fact,

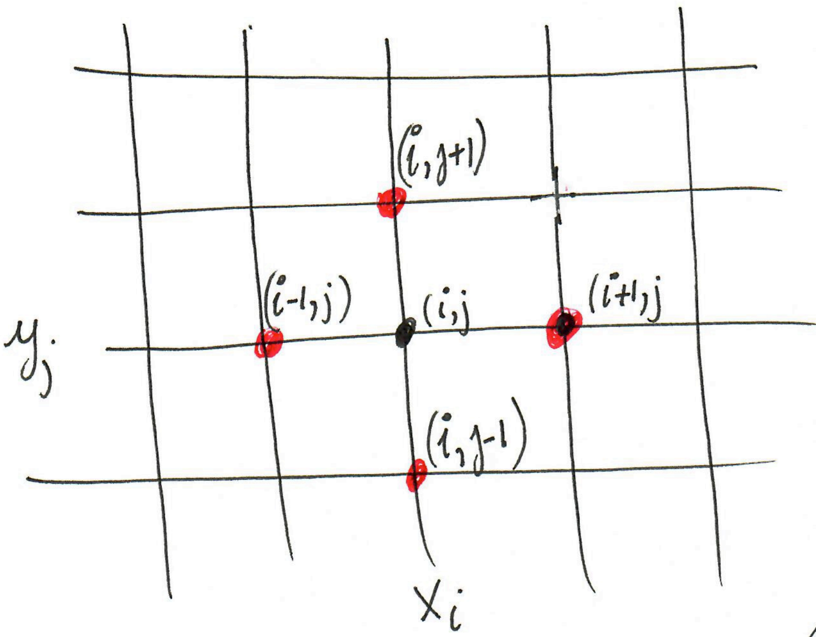
$$\boxed{\tau_{ij} = \nabla_5^2 u_{ij} - \nabla^2 u_{ij}}$$

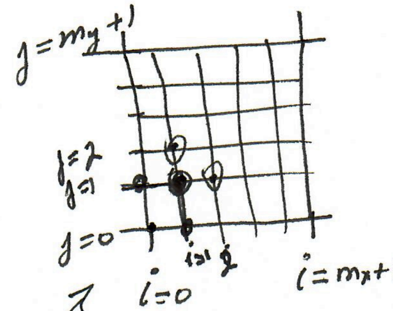
$$= \frac{1}{h^2}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{ij}\right] - \nabla^2 u_{ij}$$

$$= \cancel{\nabla^2 u_{ij}} + \frac{h^2}{12}\left[(u_{4x})_{ij} + (u_{4y})_{ij}\right] - \cancel{\nabla^2 u_{ij}} + \mathcal{O}(h^4).$$

$$= \frac{h^2}{12}\left[(u_{4x})_{ij} + (u_{4y})_{ij}\right] + \mathcal{O}(h^4)$$

Therefore, $\boxed{\tau_{ij} = \mathcal{O}(h^2).}$





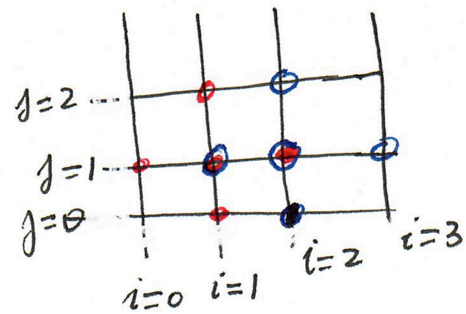
$j=1$

$i=1$: $$\underline{\underline{\theta_y v_{1,0}}} + \underline{\underline{\theta_x v_{0,1}}} - v_{1,1} + \theta_x v_{2,1} + \theta_y v_{1,2}$$

$$= \theta_f f_{1,1}$$

$i=2$: $$\underline{\underline{\theta_y v_{2,0}}} + \theta_x v_{1,1} - v_{2,1} + \theta_x v_{3,1} + \theta_y v_{2,2} = \theta_f f_{2,1}$$

$\vdots$



$\vdots$

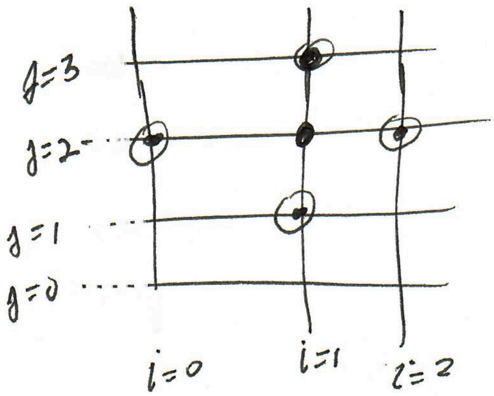
$i = m_x$:

$$\theta_y \underline{\underline{v_{m_x,0}}} + \theta_x v_{m_x-1,1} - v_{m_x,1} + \theta_x \underline{\underline{v_{m_x+1,1}}} + \theta_y v_{m_x,2}$$

$$= \theta_f f_{m_x,1}$$

$j=2$

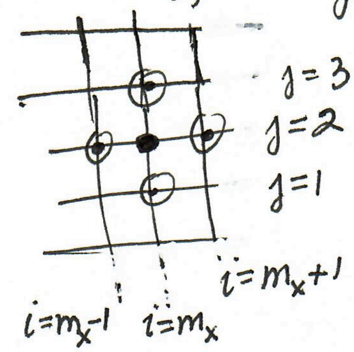
$i=1$



$$\theta_y U_{1,1} + \theta_x U_{0,2} - U_{1,2} + \theta_x U_{2,2} + \theta_y U_{1,3} = \theta_f f_{1,2}$$

$\vdots$

$i = m_x$



$$\theta_y U_{m_x,1} + \theta_x U_{m_x-1,2} - U_{m_x,2} + \theta_x U_{m_x+1,2} + \theta_y U_{m_x,3} = \theta_f f_{m_x,2}$$

$\vdots$

$j=j$

$i=1$

$\vdots$

$$i=i : \theta_y U_{i,j-1} + \theta_x U_{i-1,j} - U_{i,j} + \theta_x U_{i+1,j} + \theta_y U_{i,j+1} = \theta_f f_{i,j}$$

$\vdots$

$$
\begin{array}{l}
\\
j=1, i=1 \\
j=1, i=2 \\
j=1, i=3 \\
\vdots \\
\\
j=1, i=m_x
\end{array}
\begin{array}{c}
\overbrace{\begin{array}{cccccc}
-1 & \theta_x & 0 & 0 & \cdots & 0 \\
\theta_x & -1 & \theta_x & 0 & \cdots & 0 \\
0 & \theta_x & -1 & \theta_x & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & \theta_x & -1 & \theta_x \\
 & & & & \theta_x & -1
\end{array}}^{m_x \text{ entries}}
\end{array}
\begin{array}{c}
\overbrace{\begin{array}{cccc}
\theta_y & \cdots & & 0 \\
0 & \theta_y & 0 & 0 \\
 & & \theta_y & 0 \\
 & 0 & & \vdots \\
 & & & \\
0 & \cdots & & \theta_y
\end{array}}^{m_x \text{ entries}}
\end{array}
$$

$$
\qquad\qquad\qquad\qquad\qquad\qquad\qquad m_x \qquad\qquad\qquad\qquad\qquad\qquad (2m_x)
$$

In general, for $\underline{\underline{\text{row } j}} \neq 1$

$$
\underbrace{\begin{pmatrix}
\theta_y & & \\
0 \;\; \theta_y & & 0 \\
0 & \ddots & \\
 & & \theta_y
\end{pmatrix}}_{(m_x)\times(m_x)}
\underbrace{\begin{pmatrix}
-1 & \theta_x & 0 & 0 & \cdots & 0 \\
\theta_x & -1 & \theta_x & 0 & \cdots & 0 \\
 & \ddots & \ddots & \ddots & & 0 \\
0 & & & & \theta_x & -1
\end{pmatrix}}_{(m_x)\times(m_x)}
\underbrace{\begin{pmatrix}
\theta_y & & \\
 & \theta_y & 0 \\
0 & & \ddots \\
 & & & \theta_y
\end{pmatrix}}_{(m_x)\times(m_x)}
$$

Call

$$T = \begin{pmatrix} -1 & \theta_x & 0 & 0 & \cdots & 0 \\ \theta_x & -1 & \theta_x & 0 & \cdots & 0 \\ 0 & \theta_x & -1 & \theta_x & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ & & & \theta_x & -1 & \theta_x \\ 0 & \cdots & & 0 & \theta_x & -1 \end{pmatrix}_{m_x \times m_x}$$

$$D_y = \begin{pmatrix} \theta_y & & 0 \\ & \ddots & \\ 0 & & \theta_y \end{pmatrix},$$

then, the matix A corresponding to the finite difference Centered method is given by

$$A = \begin{array}{c} \left[\begin{matrix} T & D & 0 & 0 & \cdots & 0 \\ D & T & D & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & & \\ & & & D & T & D \\ & & & 0 & D & T \end{matrix}\right] \begin{matrix} j=1 \\ j=2 \\ \vdots \\ \vdots \\ j=m_y \end{matrix} \\ (m_x \times m_y) \times (m_x \times m_y) \end{array}$$

The unknown Vector $\vec{v}$

$$\vec{v} = \begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \\ \vdots \\ v_{m_x,1} \\ v_{1,2} \\ \vdots \\ v_{m_x,2} \\ \vdots \\ v_{1,j} \\ v_{2,j} \\ \vdots \\ v_{m_x,j} \\ \vdots \\ v_{1,m_y} \\ v_{2,m_y} \\ \vdots \\ v_{m_x,m_y} \end{bmatrix}_{[(m_x)\times(m_y)]\times 1} \qquad \vec{F} = \begin{bmatrix} \theta_f f_{11} - \theta_y v_{1,0} - \theta_x v_{0,1} \\ \theta_f f_{21} - \theta_y v_{2,0} \\ \theta_f f_{31} - \theta_y v_{3,0} \\ \vdots \\ \theta_f f_{m_x,1} - \theta_y v_{m_x,0} - \theta_x v_{m_x+1,1} \\ \theta_f f_{1,2} - \theta_x v_{0,2} \\ \vdots \ \text{No add forcing.} \\ \theta_f f_{m_x,2} - \theta_x v_{m_x+1,2} \\ \\ \theta_f f_{1,j} - \theta_x v_{0,j} \\ \theta_f f_{2,j} \ \ \text{No forcing additional.} \\ \\ \theta_f f_{m_x,j} - \theta_x v_{m_x+1,j} \\ \\ \theta_f f_{1,m_y} - \theta_y v_{1,m_y+1} - \theta_x v_{0,m_y} \\ \theta_f f_{2,m_y} - \theta_y v_{2,m_y+1} \\ \theta_f f_{m_x,m_y} - \theta_y v_{m_x,m_y+1} - \theta_x v_{m_x+1,m_y} \end{bmatrix}$$

$$\vec{F} = \begin{bmatrix} \vec{F}_1 \\ \vec{F}_2 \\ \vdots \\ \vec{F}_{m_y} \end{bmatrix}$$

$$\vec{F}_j = \theta_f \begin{bmatrix} f_{1,j} \\ f_{2,j} \\ \vdots \\ f_{m_x,j} \end{bmatrix} - \delta_{1,j}\, \theta_y \begin{bmatrix} v_{1,1} \\ v_{2,0} \\ \vdots \\ v_{m_x,0} \end{bmatrix} \quad j=1$$

$$- \delta_{m_y,j}\, \theta_y \begin{bmatrix} v_{1,m_y+1} \\ v_{2,m_y+1} \\ \vdots \\ v_{m_x,m_y+1} \end{bmatrix} \quad j=m_y$$

$$- \theta_x \begin{bmatrix} v_{0,j} \\ 0 \\ \vdots \\ 0 \\ v_{m_x,j} \end{bmatrix}$$